Gaussian quadrature formulae on the unit circle

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Abstract

Let μ be a probability measure on $[0, 2\pi]$. In this paper we shall be concerned with the estimation of integrals of the form $I_\mu(f) = (1/2\pi)\int_0^{2\pi} f(e^{i\theta})\,d\mu(\theta)$. For this purpose we will construct quadrature formulae which are exact in a certain linear subspace of Laurent polynomials. The zeros of Szegö polynomials are chosen as nodes of the corresponding quadratures. We will study this quadrature formula in terms of error expressions and convergence, as well as, its relation with certain two-point Padé approximants for the Herglotz–Riesz transform of μ. Furthermore, a comparison with the so-called Szegö quadrature formulae is presented through some illustrative numerical examples.

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1. Introduction

Let μ be a probability measure on $[0, 2\pi]$, i.e. $(1/2\pi)\int_0^{2\pi} d\mu(\theta) = 1$ and let

$$\Lambda_{-p,q} = \left\{ L(z) = \sum_{j=-p}^{q} \alpha_j z^j; \alpha_j \in \mathbb{C} \right\}, \quad p, q \in \mathbb{N}.$$

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We will deal with the approximate computation of integrals

$$I_\mu(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\mu(\theta),$$

by means of a quadrature formula such as

$$I_n(f) = \sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1} A_{m,j} f^{(j)}(\zeta_{n,m}), \quad \sum_{m=1}^{s}\alpha_m = n, \tag{1.1}$$

exact in $\Lambda_{-p,q}$, where p and q depend on n, $p \simeq q$ and are large enough. The elements of $\Lambda_{-p,q}$ are called Laurent polynomials. On the sequel Λ will denote the linear space of all Laurent polynomials, i.e., $\Lambda = \bigcup_{p,q\in\mathbb{N}} \Lambda_{-p,q}$.

First, let us consider the following inner product in the linear space $\mathbb{P}$ of polynomials with complex coefficients:

$$(f,g)_\mu = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\mu(\theta), \quad f,g\in\mathbb{P}.$$

By applying the Gram–Schmidt orthogonalization process to $\{1, z, \ldots, z^n\}$, we can build an orthogonal basis $\{\Phi_k(z)\}_{k=0}^{n}$ such that $\deg(\Phi_k) = k$ and

$$(\Phi_j, \Phi_k)_\mu = \tau_j\delta_{j,k}, \quad \tau_j > 0, \ \ 0 \leqslant j,k \leqslant n.$$

Thus, if we choose $\Phi_n(z)$ to be monic for each n, the so-called monic orthogonal polynomials on the unit circle or Szegö polynomials are obtained.

We shall use the notation $\mathbb{T} = \{z\colon |z| = 1\}$, $\mathbb{D} = \{z\colon |z| < 1\}$ and $\mathbb{E} = \{z\colon |z| > 1\}$ for the unit circle, the open unit disc and the exterior of the unit disc, respectively. $\bar{\mathbb{D}} = \mathbb{D}\cup\mathbb{T}$, $\bar{\mathbb{E}} = \mathbb{E}\cup\mathbb{T}$. Furthermore, we write $\mathbb{P}_n$ for the linear space of polynomials of degree at most n. Observe that $\Lambda_{0,n} = \mathbb{P}_n$.

If we want to construct a quadrature formula on $\mathbb{T}$, then the nodes $\zeta_{n,j}, 1 \leqslant j \leqslant n$, should lie on $\mathbb{T}$ but we know that the zeros of Φ_n lie in $\mathbb{D}$ (see [1, p. 184] or [14, Theorem 11.4.1, p. 292]). In order to construct a polynomial with zeros on the unit circle we define

$$B_n(z,\tau) = \Phi_n(z) + \tau\Phi_n^*(z), \quad |\tau| = 1, \quad \forall n\in\mathbb{N}. \tag{1.2}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is known as the reciprocal or reversed polynomial of $\Phi_n(z)$. The polynomial $B_n(z,\tau)$ has n simple zeros on $\mathbb{T}$ interlacing with the zeros of $B_{n-1}(z,\tau)$ (see [8]). The system $\{B_n(z,\tau)\}_n$ is said to be a sequence of para-orthogonal polynomials with respect to μ. If $\epsilon_{n,j}$, $1 \leqslant j \leqslant n$, are the zeros of $B_n(z,\tau)$ then the quadrature formula $I_n(f)$ given by formula (1.1) is exact in $\Lambda_{-(n-1),n-1}$ and the coefficients $A_{n,j}, 1 \leqslant j \leqslant n$ are all positive [8]. In this case, $I_n(f)$ is called the n-point Szegö quadrature formula. Furthermore, $\Lambda_{-(n-1),n-1}$ is the maximum domain of validity in the sense that there cannot exist an n-point Szegö quadrature formula that is valid for all $f\in\Lambda_{-n,n-1}$ or for all $f\in\Lambda_{-(n-1),n}$ (see [8]).

In this paper we will see how to construct certain quadrature formulae like (1.1) which are exact in $\Lambda_{-(n-1),n}$ (a similar treatment can also be given for exactness in $\Lambda_{-n,n-1}$). In this respect, the zeros of Szegö polynomials are to be used as nodes of the corresponding quadrature formulae. Since we need to evaluate the integrand $f(z)$ in points of $\mathbb{D}$ (as well as its derivatives), we are forced to deal with functions f analytic in $\mathbb{D}$.

The paper is organized as follows. In Section 2 we will introduce the Gaussian quadrature formulae. In Section 3 an alternative way to compute the coefficients of such Gaussian formulae is given. In Section 4 we study its relation to interpolatory quadrature formulae and certain two-point Padé approximants for the Herglotz–Riesz transform of μ. In Section 5 we give error expressions and some convergence results. Finally, in Section 6 we compare the Gaussian formulae with the Szegö formulae through some illustrative examples.

2. Gaussian formulae

Note that, in the n-point Szegö quadrature formula defined above, there are $2n$ parameters, namely, the n nodes and the n coefficients but the dimension of the space $\Lambda_{-(n-1),n-1}$ is $2n-1$. Therefore, we shall be concerned with quadrature formulae on $\mathbb{T}$ with the maximum degree of precision. For this reason we will call them, Gaussian quadrature formulae on the unit circle.

It is well known, (see [6, p. 198]), that

$$\frac{1}{2\pi}\int_0^{2\pi} P(x)\,\mathrm{d}\mu(\theta) = \frac{k_n}{2\pi}\int_0^{2\pi} \frac{P(x)}{|\Phi_n(x)|^2}\,\mathrm{d}\theta, \quad x = \mathrm{e}^{\mathrm{i}\theta} \tag{2.1}$$

for all $P \in \mathbb{P}_n$ where $k_n = (\Phi_n(z), \Phi_n(z))_\mu$.

Thus, we define

$$I_n(f) = \frac{k_n}{2\pi}\int_0^{2\pi} \frac{f(x)}{|\Phi_n(x)|^2}\,\mathrm{d}\theta, \quad x = \mathrm{e}^{\mathrm{i}\theta} \tag{2.2}$$

as well as

$$I_\mu(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}\mu(\theta), \quad x = \mathrm{e}^{\mathrm{i}\theta},\ f \in \Lambda.$$

We will study this quadrature formula in terms of error expressions and convergence, as well as, its relation to certain two-point Padé approximants for the Herglotz–Riesz Transform of μ. Also, a comparison with the n-point Szegö quadrature formula is presented. Note that $I_n \xrightarrow{*} I_\mu$, [5], i.e., $I_n(f) \to I_\mu(f)$ for all continuous functions f on $\mathbb{T}$.

As we have stated above, the zeros of the monic Szegö polynomial with respect to μ, $\Phi_n(z)$, lie in $\mathbb{D}$. Thus, we can write $\Phi_n(z) = z^\alpha \prod_{m=1}^{s}(z - x_{n,m})^{\alpha_m}$ with $\alpha = \alpha(n)$, $0 \leqslant \alpha \leqslant n$, $\sum_{m=1}^{s} \alpha_m = n - \alpha$ and $x_{n,m} \neq 0, \forall m = 1, \ldots, s$.

First, consider the case $\alpha = n$. Then $\Phi_n(z) = z^n$. Thus, for any j: $1 \leqslant j \leqslant n$ and $x = \mathrm{e}^{\mathrm{i}\theta}$, we get

$$I_\mu(x^j) = \frac{1}{2\pi}\int_0^{2\pi} x^j\,\mathrm{d}\mu(\theta) = \frac{k_n}{2\pi}\int_0^{2\pi} \frac{x^j}{|\Phi_n(x)|^2}\,\mathrm{d}\theta = \frac{k_n}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}j\theta}\,\mathrm{d}\theta = 0.$$

Therefore, $\overline{I_\mu(x^j)} = I_\mu(x^{-j}) = 0; 1 \leqslant j \leqslant n$.

On the other hand, for $j \geqslant 1$,

$$I_n(x^{-j}) = \frac{k_n}{2\pi\mathrm{i}}\int_{\mathbb{T}} \frac{1}{x^{j+1}}\,\mathrm{d}x = 0.$$

For $j=0$ we have that $I_\mu(1)=I_n(1)=1$ and consequently, if $L\in\Lambda_{-n,n}$, namely $L(z)=\sum_{j=-n}^{n}\beta_j z^j$, then $I_\mu(L)=I_n(L)=\beta_0$.

This case will be excluded in the rest of our work. Thus,

$$\Phi_n(z)=z^\alpha\tilde{\Phi}_{n-\alpha}(z), \quad 0\leqslant\alpha\leqslant n-1, \tag{2.3}$$

where

$$\tilde{\Phi}_{n-\alpha}(z)=\prod_{m=1}^{s}(z-x_m)^{\alpha_m}, \quad 0<|x_m|<1 \text{ and } \sum_{m=1}^{s}\alpha_m=n-\alpha. \tag{2.4}$$

In order to characterize the nodal polynomial of our quadrature formulae, we have, as a first result, the following

Proposition 2.1. *Let α be a nonnegative integer such that $0\leqslant\alpha\leqslant n-1$ and let*

$$I_n(f)=\sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1}A_{m,j}f^{(j)}(x_m), \quad x_m\neq 0, \ m=1,\dots,s$$

be an exact quadrature formula in $\Lambda_{-(k-\alpha),2n-k-1}, 0\leqslant k\leqslant 2n$. Then, the polynomial

$$\sigma_n(z)=z^\alpha\prod_{m=1}^{s}(z-x_m)^{\alpha_m},$$

where $\sum_{m=1}^{s}\alpha_m=n-\alpha$, satisfies the following orthogonality properties with respect to the positive measure μ:

$$(\sigma_n(z),z^p)_\mu=0 \quad \text{for all } k-n+1\leqslant p\leqslant k, \ 0\leqslant k\leqslant 2n. \tag{2.5}$$

Proof.

$$(\sigma_n(z),z^p)_\mu=\int_0^{2\pi}\sigma_n(\mathrm{e}^{\mathrm{i}\theta})\overline{\mathrm{e}^{\mathrm{i}p\theta}}\,\mathrm{d}\mu(\theta)=\int_0^{2\pi}\frac{1}{\mathrm{e}^{\mathrm{i}(p-\alpha)\theta}}\prod_{m=1}^{s}(\mathrm{e}^{\mathrm{i}\theta}-x_m)^{\alpha_m}\,\mathrm{d}\mu(\theta).$$

Let $\tilde{\sigma}_{n-\alpha}(z)=\prod_{m=1}^{s}(z-x_m)^{\alpha_m}$, then, since $I_n(f)$ is exact in $\Lambda_{-(k-\alpha),2n-k-1}$, for all $k-n+1\leqslant p\leqslant k$, $0\leqslant k\leqslant 2n$, one has

$$\begin{aligned}(\sigma_n(z),z^p)_\mu&=\sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1}A_{m,j}\frac{\mathrm{d}^j}{\mathrm{d}z^j}(\tilde{\sigma}_{n-\alpha}(z)z^{-(p-\alpha)})(x_m)\\&=\sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1}A_{m,j}\left(\sum_{l=0}^{j}\binom{j}{l}(z^{-(p-\alpha)})^{(j-l)}(x_m)\tilde{\sigma}_{n-\alpha}^{(l)}(x_m)\right).\end{aligned}$$

Since x_m is a zero of $\tilde{\sigma}_{n-\alpha}$ with multiplicity α_m for all $m=1,\dots,s$, then $\tilde{\sigma}_n^{(l)}(x_m)=0, \forall l=0,\dots,j, \ j=0,\dots,\alpha_m-1$. Thus,

$$(\sigma_n(z),z^p)_\mu=0, \quad \forall k-n+1\leqslant p\leqslant k. \qquad \square$$

In particular, for $k = n-1$, $I_n(f)$ is exact in $\Lambda_{-(n-\alpha-1),n}$ and then $\sigma_n \equiv \Phi_n$.

On the other hand, if $k = n$, $I_n(f)$ is exact in $\Lambda_{-(n-\alpha),n-1}$ and we have that $\sigma_n \equiv \Phi_n^*$, where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$.

In order to calculate the coefficients in the quadrature formula, which is exact in $\Lambda_{-(n-\alpha-1),n}$, consider first the case where the function f is analytic in $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$. Then, we have

$$|\Phi_n(x)|^2 = \Phi_n(x)\overline{\Phi_n(x)} = \Phi_n(x)\frac{\Phi_n^*(x)}{x^n},$$

where $\Phi_n^*(x) = x^n\overline{\Phi_n(1/\bar{x})}$. For $x \in \mathbb{T}$ one has $\Phi_n^*(x) = x^n\overline{\Phi_n(x)}$. Consider $x = \mathrm{e}^{\mathrm{i}\theta}$. Thus, by applying the Residue Theorem, one has

$$\begin{aligned}
I_n(f) &= \frac{k_n}{2\pi}\int_0^{2\pi}\frac{f(x)x^n}{\Phi_n(x)\Phi_n^*(x)}\,\mathrm{d}\theta = \frac{k_n}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{f(x)x^{n-1}}{\Phi_n(x)\Phi_n^*(x)}\,\mathrm{d}x = \frac{k_n}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{f(x)x^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(x)\Phi_n^*(x)}\,\mathrm{d}x \\
&= k_n\sum_{m=1}^{s}\mathrm{Res}(z = x_m) = k_n\sum_{m=1}^{s}\frac{1}{(\alpha_m-1)!}\lim_{z\to x_m}\frac{\mathrm{d}^{\alpha_m-1}}{\mathrm{d}z^{\alpha_m-1}}\left(\frac{(z-x_m)^{\alpha_m}z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)}f(z)\right) \\
&= k_n\sum_{m=1}^{s}\frac{1}{(\alpha_m-1)!}\lim_{z\to x_m}\sum_{j=0}^{\alpha_m-1}\binom{\alpha_m-1}{j}\frac{\mathrm{d}^{\alpha_m-1-j}}{\mathrm{d}x^{\alpha_m-1-j}}\left(\frac{(z-x_m)^{\alpha_m}z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)}\right)f^{(j)}(z) \\
&= \sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1}\left(\frac{k_n}{(\alpha_m-1-j)!j!}\lim_{z\to x_m}\frac{\mathrm{d}^{\alpha_m-1-j}}{\mathrm{d}z^{\alpha_m-1-j}}\left(\frac{(z-x_m)^{\alpha_m}z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)}\right)\right)f^{(j)}(x_m).
\end{aligned}$$

Therefore,

$$I_n(f) = \sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1}A_{m,j}f^{(j)}(x_m), \tag{2.6}$$

where

$$A_{m,j} = \frac{k_n}{(\alpha_m-1-j)!j!}\lim_{z\to x_m}\frac{\mathrm{d}^{\alpha_m-1-j}}{\mathrm{d}z^{\alpha_m-1-j}}\left(\frac{(z-x_m)^{\alpha_m}z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)}\right). \tag{2.7}$$

In particular, if all the zeros are simple and $\alpha = 0$, then the coefficients are given by

$$A_j = \frac{k_n x_j^{n-1}}{\Phi_n'(x_j)\Phi_n^*(x_j)}, \quad j = 1, \ldots, n. \tag{2.8}$$

Remark 2.2. If $\alpha \neq 0$, formula (2.6) can be carried for a function $f(z)$ meromorphic in $\mathbb{D}$, continuous on $\mathbb{T}$ so that $z = 0$ is the only pole with a certain multiplicity.

Remark 2.3. Note that $I_n(f)$ defined as in (2.2) can be computed for every integrable function f on $\mathbb{T}$. However, one obtains a formula like (1.1) only when the function is analytic on $\mathbb{D}$. We are interested in this discretization since otherwise we would have to calculate integral (2.2).

Suppose now that f is analytic in $\mathbb{E}$ and continuous on $\bar{\mathbb{E}}$. Since

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})} = z^n \left(\frac{1}{z^\alpha} \prod_{m=1}^{s} \left(\frac{1}{z} - \bar{x}_m \right)^{\alpha_m} \right) = \prod_{m=1}^{s} (1 - z\bar{x}_m)^{\alpha_m},$$

the zeros of $\Phi_n^*(z)$ are $z_m = 1/\bar{x}_m$, $m = 1, \ldots, s$ with multiplicity α_m, $m = 1, \ldots, s$. Therefore,

$$\begin{aligned} I_n(f) &= \frac{k_n}{2\pi} \int_0^{2\pi} \frac{f(x)x^n}{\Phi_n(x)\Phi_n^*(x)} \, \mathrm{d}\theta = \frac{k_n}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{f(x)x^{n-1}}{\Phi_n(x)\Phi_n^*(x)} \, \mathrm{d}x = \frac{k_n}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{f(x)x^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(x)\Phi_n^*(x)} \, \mathrm{d}x \\ &= k_n \sum_{m=1}^{s} \mathrm{Res}(z = z_m) = k_n \sum_{m=1}^{s} \frac{1}{(\alpha_m - 1)!} \lim_{z \to z_m} \frac{\mathrm{d}^{\alpha_m - 1}}{\mathrm{d}z^{\alpha_m - 1}} \left(\frac{(z - z_m)^{\alpha_m} z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)} f(z) \right) \\ &= k_n \sum_{m=1}^{s} \frac{1}{(\alpha_m - 1)!} \lim_{z \to z_m} \sum_{j=0}^{\alpha_m - 1} \binom{\alpha_m - 1}{j} \frac{\mathrm{d}^{\alpha_m - 1 - j}}{\mathrm{d}z^{\alpha_m - 1 - j}} \left(\frac{(z - z_m)^{\alpha_m} z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)} \right) f^{(j)}(z) \\ &= \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m - 1} \left(\frac{k_n}{(\alpha_m - 1 - j)! j!} \lim_{z \to z_m} \frac{\mathrm{d}^{\alpha_m - 1 - j}}{\mathrm{d}z^{\alpha_m - 1 - j}} \left(\frac{(z - z_m)^{\alpha_m} z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)} \right) \right) f^{(j)}(z_m). \end{aligned}$$

Therefore, the coefficients are given by

$$A_{m,j} = \frac{k_n}{(\alpha_m - 1 - j)! j!} \lim_{z \to z_m} \frac{\mathrm{d}^{\alpha_m - 1 - j}}{\mathrm{d}z^{\alpha_m - 1 - j}} \left(\frac{(z - z_m)^{\alpha_m} z^{n-\alpha-1}}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)} \right).$$

Since we are interested in functions f which are analytic in $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$, hereafter the study will be focused in the quadrature formula $I_n(f)$ given by (2.6) which will be called, as already indicated, Gaussian quadrature formula on the unit circle.

Proposition 2.4. *A quadrature formula $I_n(f)$ is exact in $\Lambda_{-(n-\alpha-1),n}$ if and only if the coefficients are given as in formula* (2.7) *and the nodes are the nonvanishing zeros of the monic Szegö polynomial $\Phi_n(z)$ with respect to μ, where $\Phi_n(z) = z^\alpha \tilde{\Phi}_{n-\alpha}(z), \tilde{\Phi}_{n-\alpha}(0) \neq 0$.*

Proof. $\Rightarrow$: If $I_n(f)$ is exact in $\Lambda_{-(n-\alpha-1),n}$ then, we have already proved that the coefficients $A_{m,j}$, $m = 1, \ldots, s$, $j = 0, \ldots, \alpha_m - 1$ are given as in formula (2.7). By Proposition 2.1 ($k = n - 1$), we know that the nodes x_m, $m = 1, \ldots, s$ are the nonvanishing zeros of the monic Szegö polynomial with respect to μ.

$\Leftarrow$: For $0 \leqslant p \leqslant n$, from formula (2.1), we get

$$\frac{1}{2\pi} \int_0^{2\pi} x^p \, \mathrm{d}\mu(\theta) = \frac{k_n}{2\pi} \int_0^{2\pi} \frac{x^p}{|\Phi_n(x)|^2} \, \mathrm{d}\theta \quad (x = \mathrm{e}^{\mathrm{i}\theta}).$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} x^{-p} \, \mathrm{d}\mu(\theta) = \frac{k_n}{2\pi} \int_0^{2\pi} \frac{x^{-p}}{|\Phi_n(x)|^2} \, \mathrm{d}\theta, \quad 1 \leqslant p \leqslant n.$$

Let $1 \leqslant p \leqslant n-\alpha-1$. Since $1 \leqslant \alpha \leqslant n-1$, one has

$$I_\mu(x^{-p}) = \frac{1}{2\pi}\int_0^{2\pi} x^{-p}\,\mathrm{d}\mu(\theta) = \frac{k_n}{2\pi}\int_0^{2\pi}\frac{x^{-p}}{|\Phi_n(x)|^2}\,\mathrm{d}\theta$$
$$= \frac{k_n}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{x^{n-p-1}}{\Phi_n(x)\Phi_n^*(x)}\,\mathrm{d}x = \frac{k_n}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{x^{n-p-\alpha}}{x\tilde{\Phi}_{n-\alpha}(x)\Phi_n^*(x)}\,\mathrm{d}x$$
$$= k_n\,\mathrm{Res}(z=0) + k_n\sum_{m=1}^{s}\mathrm{Res}(z=x_m) = k_n\,\mathrm{Res}(z=0) + I_n(x^{-p}).$$

But, in this case $\mathrm{Res}(z=0)=0$, so $I_\mu(x^{-p}) = I_n(x^{-p})$. Let $n-\alpha \leqslant p \leqslant n$. Then, again we get

$$I_\mu(x^{-p}) = k_n\,\mathrm{Res}(z=0) + I_n(x^{-p}).$$

Let $q = p-n+\alpha+1$. Then $1 \leqslant q \leqslant \alpha$ and

$$\mathrm{Res}(z=0) = \frac{1}{(q-1)!}\lim_{z\to 0}\frac{\mathrm{d}^{q-1}}{\mathrm{d}z^{q-1}}\left(\frac{1}{\tilde{\Phi}_{n-\alpha}(z)\Phi_n^*(z)}\right) \neq 0. \qquad \square$$

3. Calculation of the coefficients

We have shown that the coefficients of the quadrature formula $I_n(f)$ are given as in formula (2.7). Here we give an alternative method for their computation.

Let

$$L_l^p(z) = \frac{\Phi_n(z)}{(z-x_p)^{\alpha_p-l}}, \quad l=0,\ldots,\alpha_p-1, \quad p=1,\ldots,m. \tag{3.1}$$

Since $L_l^p \in \Lambda_{-(n-1),n}$ one has

$$\frac{1}{2\pi}\int_0^{2\pi} L_l^p(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) = \sum_{k=1}^{m}\sum_{j=0}^{\alpha_k-1} A_{k,j}\lim_{z\to x_k}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\left(\frac{\Phi_n(z)}{(z-x_p)^{\alpha_p-l}}\right),$$
$$l=0,\ldots,\alpha_p-1, \quad p=1,\ldots,m.$$

Let

$$M_j^q = \lim_{z\to x_k}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\left(\frac{\Phi_n(z)}{(z-x_k)^q}\right),$$

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j}\left(\frac{\Phi_n(z)}{(z-x_k)^q}\right) = \sum_{l=0}^{j}\binom{j}{l}\frac{\mathrm{d}^{j-l}}{\mathrm{d}z^{j-l}}\left((z-x_k)^{-q}\right)\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left(\Phi_n(z)\right)$$
$$= \sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}(q)_{j-l}(z-x_k)^{-(q+j-l)}\Phi_n^{(l)}(z),$$

where $(q)_{j-l} = q(q+1)\dots(q+j-l-1)$ is the symbol of Pochhammer. Thus,

$$\begin{aligned}
M_j^q &= \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} (q)_{j-l} \lim_{z\to x_k} \frac{\Phi_n^{(l)}(z)}{(z-x_k)^{q+j-l}} \\
&= \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} (q)_{j-l} \frac{\Phi_n^{(q+j)}(x_k)}{(q+j-l)!} \\
&= \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \frac{\Phi_n^{(q+j)}(x_k)}{(q+j-l)(q-1)!} \\
&= \frac{\Phi_n^{(q+j)}(x_k)}{(q-1)!} \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \frac{1}{q+j-l}.
\end{aligned}$$

Therefore, we have

$$M_j^q = \frac{\Phi_n^{(q+j)}(x_k)}{(q-1)!} \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \frac{1}{q+j-l}. \tag{3.2}$$

For all the coefficients of the form $A_{k,\alpha_k-1}, \forall k = 1,\dots,m$, we obtain

$$A_{k,\alpha_k-1} = \frac{1}{M_{\alpha_k-1}^1} \frac{1}{2\pi} \int_0^{2\pi} L_{\alpha_k-1}^k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta), \quad \forall k = 1,\dots,m.$$

For the coefficients $A_{k,\alpha_k-2}, \forall k = 1,\dots,m$, one has

$$M_{\alpha_k-2}^2 A_{k,\alpha_k-2} + M_{\alpha_k-1}^2 A_{k,\alpha_k-1} = \frac{1}{2\pi} \int_0^{2\pi} L_{\alpha_k-2}^k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta), \quad \forall k = 1,\dots,m.$$

Therefore,

$$A_{k,\alpha_k-2} = \frac{1}{M_{\alpha_k-2}^2} \left(\frac{1}{2\pi} \int_0^{2\pi} L_{\alpha_k-2}^k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) - M_{\alpha_k-1}^2 A_{k,\alpha_k-1} \right), \quad \forall k = 1,\dots,m.$$

For $A_{k,\alpha_k-3}, \forall k = 1,\dots,m$, we get

$$\begin{aligned}
&M_{\alpha_k-3}^3 A_{k,\alpha_k-3} + M_{\alpha_k-2}^3 A_{k,\alpha_k-2} + M_{\alpha_k-1}^3 A_{k,\alpha_k-1} \\
&\quad = \frac{1}{2\pi} \int_0^{2\pi} L_{\alpha_k-3}^k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta), \quad \forall k = 1,\dots,m.
\end{aligned}$$

So,

$$\begin{aligned}
A_{k,\alpha_k-3} &= \frac{1}{M_{\alpha_k-3}^3} \left(\frac{1}{2\pi} \int_0^{2\pi} L_{\alpha_k-3}^k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) - (M_{\alpha_k-2}^3 A_{k,\alpha_k-2} + M_{\alpha_k-1}^3 A_{k,\alpha_k-1}) \right), \\
&\quad \forall k = 1,\dots,m.
\end{aligned}$$

Thus, in general

$$A_{k,j} = \frac{1}{M_j^{\alpha_k - j}} \left(\frac{1}{2\pi} \int_0^{2\pi} L_j^k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) - \sum_{r=j+1}^{\alpha_k - 1} M_r^{\alpha_k - j} A_{k,r} \right),$$

$$\forall j = 0, \ldots, \alpha_k - 1, \ \ \forall k = 1, \ldots, m.$$

If all the zeros are simple, i.e., if $\alpha_k = 1, \forall k = 1, \ldots, m$, then from formula (3.1) we have $L_0^p(z) = \Phi_n(z)/(z - x_p)$ and from formula (2.7)

$$A_{k,0} = \frac{1}{M_0^1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi_n(z)}{z - x_k}\,\mathrm{d}\mu(\theta) = \frac{k_n}{M_0^1} \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \frac{z^{n-1}}{(z - x_k)\Phi_n^*(z)}\,\mathrm{d}z.$$

Using the Residue Theorem, we get

$$A_{k,0} = \frac{k_n}{M_0^1} \frac{x_k^{n-1}}{\Phi_n^*(x_k)}.$$

From formula (3.2) $M_0^1 = \Phi_n'(x_k)$ and so we come back to formula (2.8). The integrals $1/2\pi \int_0^{2\pi} L_l^p(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta)$ can be computed in the following way:

$$\frac{1}{2\pi} \int_0^{2\pi} L_l^p(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) = \frac{k_n}{2\pi\mathrm{i}} \int_{\mathbb{T}} \frac{L_l^p(z) z^{n-1}}{\Phi_n(z)\Phi_n^*(z)}\,\mathrm{d}z$$

$$= \frac{k_n}{2\pi\mathrm{i}} \int_{\mathbb{T}} \frac{z^{n-1}}{(z - z_p)^{\alpha_p - l}\Phi_n^*(z)}\,\mathrm{d}z.$$

Using the Residue Theorem, one has

$$\frac{1}{2\pi} \int_0^{2\pi} L_l^p(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) = \frac{k_n}{(\alpha_p - l - 1)!} \lim_{z \to x_p} \frac{\mathrm{d}^{\alpha_p - l - 1}}{\mathrm{d}z^{\alpha_p - l - 1}} \left(\frac{z^{n-1}}{\Phi_n^*(z)} \right).$$

4. Interpolatory quadrature formulae and two-point Padé approximants

As we have mentioned in the Introduction, in this section we will deal with the relation between the so-called interpolatory quadrature formulae and certain two-point Padé approximants for the Herglot–Riesz transform of the measure μ. This will allow us to give some results about convergence and estimates of the rate of convergence for the Gaussian quadratures as done in Section 5.

4.1. Interpolatory quadrature formulae

Let $\Phi_n(z)$ be the nth monic Szegö polynomial with respect to μ. We have seen that we can write $\Phi_n(z)$ as in (2.3) satisfying the conditions given by (2.4). Since the zeros $\{x_m\}_{m=1}^s$ of $\tilde{\Phi}_{n-\alpha}(z)$ are all different from zero and $\{z^j\}_{j=-p}^q$ $(p + q = n - \alpha - 1)$ is a Chebyshev system on any set $A \subset \mathbb{C}$ such that $0 \notin A$, the coefficients $A_{m,j}$; $0 \leqslant j \leqslant \alpha_m - 1$, $m = 1, \ldots, s$ can be uniquely determined so that the corresponding quadrature formula $I_n(f) = \sum_{m=1}^s \sum_{j=0}^{\alpha_m - 1} A_{m,j} f^{(j)}(x_m)$ satisfies $I_\mu(f) = I_n(f)$

for all $f \in \Lambda_{-p,q}$, $(p+q=n-\alpha-1)$. On the other hand, starting from the nodes $\{x_m\}_{m=1}^{s}$, it can be proved that there exists a unique Laurent polynomial $L_n(z) \in \Lambda_{-p,q}$ such that

$$L_n^{(j)}(x_m) = f^{(j)}(x_m), \quad j = 0, \ldots, \alpha_m - 1, \ m = 1, \ldots, s.$$

Furthermore, one can write $L_n(z) = \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m - 1} \tilde{L}_{m,j}(z) f^{(j)}(x_m)$, where $\tilde{L}_{m,j} \in \Lambda_{-p,q}$ and

$$\tilde{L}_{m,j}^{(l)}(x_k) = \delta_{m,k}\delta_{j,l}, \quad l, j = 0, \ldots, \alpha_m - 1, \ m, k = 1, \ldots, s. \tag{4.1}$$

Here, and as usual, for any nonnegative integers j and k, $\delta_{j,k}$ represents the well-known Kronecker's delta, i.e.,

$$\delta_{j,k} = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Now, proceeding as in [10, p. 80] it follows

$$I_\mu(L_n(z)) = \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m - 1} I_\mu(\tilde{L}_{m,j}(z)) f^{(j)}(x_m) = I_n(f), \quad j = 0, \ldots, \alpha_m - 1, \ m = 1, \ldots, s,$$

where $I_\mu(\tilde{L}_{m,j}(z)) = A_{m,j}$, $j = 0, \ldots, \alpha_m - 1$, $m = 1, \ldots, s$. For this reason, the quadrature formula $I_n(f)$ defined above is said to be an interpolatory type quadrature formula in $\Lambda_{-p,q}$. In parallel with the polynomial case (see [10, p. 101]) a connection between interpolatory quadrature formulae and Gaussian formulae, as defined in Section 2, is given in the following

Theorem 4.1. *Let α be an integer such that $0 \leqslant \alpha \leqslant n-1$. Then the quadrature formula*

$$I_n(f) = \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m - 1} A_{m,j} f^{(j)}(x_m), \quad x_m \neq 0, \ m = 1, \ldots, s$$

is exact in $\Lambda_{-(n-1-\alpha),n}$ if and only if

(i) *$I_n(f)$ is of interpolatory type in $\Lambda_{-p,q}$ where p and q are arbitrary nonnegative integers such that $p+q=n-\alpha-1$.*

(ii) *The nodes $\{x_m\}_{m=1}^{s}$ with multiplicity $\{\alpha_m\}_{m=1}^{s}$, $\sum_{m=1}^{s} \alpha_m = n - \alpha$, are the nonvanishing zeros of $\Phi_n(z)$, i.e., the zeros of $\tilde{\Phi}_{n-\alpha}(z)$.*

Proof. $\Rightarrow$: Condition (i) is trivial and (ii) is already proved.

$\Leftarrow$: Assume that $I_n(f) = \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m - 1} A_{m,j} f^{(j)}(x_m)$ is exact in $\Lambda_{-p,q}$ $(p+q=n-\alpha-1)$.

For $L \in \Lambda_{-(n-\alpha-1),n}$ write

$$R(z) = L(z) - \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m - 1} \tilde{L}_{m,j}(z) L^{(j)}(x_m) \in \Lambda_{-(n-\alpha-1),n},$$

where $\tilde{L}_{m,j} \in \Lambda_{-p,q}$ satisfies (4.1).

$R \in \Lambda_{-(n-\alpha-1),n}$ and $R^{(i)}(x_m) = 0,\ i = 0,\dots,\alpha_m - 1,\ m = 1,\dots,s$. Therefore, one has

$$R(z) = \frac{S(z)\tilde{\Phi}_{n-\alpha}(z)}{z^{n-\alpha-1}} = \frac{P(z)}{z^{n-\alpha-1}}, \quad P \in \mathbb{P}_{2n-\alpha-1},\ S \in \mathbb{P}_{n-1}.$$

$L(z) = R(z) + \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m-1} \tilde{L}_{m,j}(z) L^{(j)}(x_m)$. Thus, since $I_\mu(\tilde{L}_{m,j}(z)) = A_{m,j},\ j = 0,\dots,\alpha_m - 1,\ m = 1,\dots,s$,

$$I_\mu(L) = I_\mu(R) + \sum_{m=1}^{s} \sum_{j=0}^{\alpha_m-1} I_\mu(\tilde{L}_{m,j}(z)) L^{(j)}(x_m) = I_\mu(R) + I_n(L).$$

Now, set $x = \mathrm{e}^{\mathrm{i}\theta}$

$$\begin{aligned} I_\mu(R) &= \frac{1}{2\pi}\int_0^{2\pi} R(x)\,\mathrm{d}\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{S(x)\tilde{\Phi}_{n-\alpha}(x)}{x^{n-\alpha-1}}\,\mathrm{d}\mu(\theta) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{x^\alpha S(x)\tilde{\Phi}_{n-\alpha}(x)}{x^{n-1}}\,\mathrm{d}\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{S(x)\Phi_n(x)}{x^{n-1}}\,\mathrm{d}\mu(\theta) = 0 \end{aligned}$$

because of the orthogonality of $\Phi_n(z)$.

Now, it remains to prove that $I_n(f)$ does not depend on the election of the integer p. But, this fact immediately follows from Proposition 2.3 and formula (2.7). $\square$

4.2. Two-point Padé approximants

In this subsection we will study the relation between the Gaussian formulae and certain two-point Padé approximants for the Herglotz–Riesz transform: For every $z \notin \mathbb{T}$, the Herglotz–Riesz transform of the measure μ, is given by

$$F_\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{x+z}{x-z}\,\mathrm{d}\mu(\theta), \quad x = \mathrm{e}^{\mathrm{i}\theta}. \tag{4.2}$$

It is very well known (see [7, Theorem 13.1, p. 20]) that the following series expansions around 0 and ∞ hold:

$$L_0(z) = \mu_0 + 2\sum_{j=1}^{\infty} \mu_j z^j, \quad |z| < 1,$$

$$L_\infty(z) = -\mu_0 - 2\sum_{j=1}^{\infty} \mu_{-j} z^{-j}, \quad |z| > 1$$

with $\mu_k = (1/2\pi)\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\mu(\theta); k \in \mathbb{Z}$.

Let

$$\Omega_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{z+x}{z-x}\left(\Phi_n(x) - \Phi_n(z)\right)\right)\mathrm{d}\mu(\theta), \quad (x = \mathrm{e}^{\mathrm{i}\theta}), \tag{4.3}$$

where $\Phi_n(z)$ denotes the Szegö polynomial with respect to the measure μ. Ω_n is called the associated Szegö polynomial with Φ_n. Note that there is a difference in the notation of such polynomial Ω_n

between the one given in [8] and the traditional one in the theory of orthogonal polynomials (see, e.g., [7]). Then, it holds (see [8]) that the rational function $\Omega_n(z)/\Phi_n(z)$ satisfies

$$\Phi_n(z)L_0(z)-\Omega_n(z)=O(z^n),\quad \Phi_n(z)L_\infty(z)-\Omega_n(z)=O\left(\frac{1}{z}\right).$$

In other words, this rational function represents the weak (n,n) two-point Padé approximant of an order $(n,n+1)$ for the pair (L_0,L_∞), (see [8]).

If $\Phi_n(0)\neq 0$, i.e., $\alpha=0$, then it can be shown that

$$L_0(z)-\frac{\Omega_n(z)}{\Phi_n(z)}=O(z^n),\quad L_\infty(z)-\frac{\Omega_n(z)}{\Phi_n(z)}=O\left(\left(\frac{1}{z}\right)^{n+1}\right).$$

In such a case this rational function represents the (n,n) two-point Padé approximant of an order $(n,n+1)$ for the pair (L_0,L_∞).

Now, let us see how these approximants can be deduced from the Gaussian formulae. Indeed, taking into account the function in the variable x (z being a parameter)

$$h(x,z)=\frac{x+z}{x-z},\quad z\in\mathbb{E}, \tag{4.4}$$

which is analytic in $\bar{\mathbb{D}}$, applying our quadrature formula $I_n(f)$, which is exact in $\Lambda_{-(n-\alpha-1),n}$, we get

$$I_n(h(\bullet,z))=\frac{\tilde{Q}_n(z)}{P_n(z)}, \tag{4.5}$$

where $P_n(z)=\prod_{m=1}^{s}(z-x_m)^{\alpha_m}=\tilde{\Phi}_{n-\alpha}(z)$ and $\tilde{Q}_n(z)$ is a polynomial of degree at most $n-\alpha$.

Let $R_n(x,z)\in\Lambda_{-p,q}$ (z being a parameter) be the Laurent polynomial interpolating $h(x,z)$ at the nodes $\{x_j\}_{m=1}^{s}$, with multiplicity α_m, respectively ($m=1,\ldots,s$; $\sum_{m=1}^{s}\alpha_m=n-\alpha$ and $p+q=n-\alpha-1$). Then, by Theorem 4.1 one has

$$\frac{\tilde{Q}_n(z)}{\tilde{\Phi}_{n-\alpha}(z)}=I_\mu(R_n(\bullet,z)).$$

Now, it can be easily checked that $R_n(x,z)$ is given by

$$R_n(x,z)=1+\frac{2z}{x-z}\left(1-\frac{z^p\tilde{\Phi}_{n-\alpha}(x)}{x^p\tilde{\Phi}_{n-\alpha}(z)}\right). \tag{4.6}$$

From (4.6) we can obtain the following integral representation for the polynomial $\tilde{Q}_n(z)$, namely:

$$\tilde{Q}_n(z)=I_\mu\left\{\tilde{\Phi}_{n-\alpha}(z)+\frac{2z}{x-z}\left(\tilde{\Phi}_{n-\alpha}(z)-\frac{z^p}{x^p}\tilde{\Phi}_{n-\alpha}(x)\right)\right\},$$
$$0\leqslant p\leqslant n-\alpha-1. \tag{4.7}$$

After some elementary calculations and having in mind the orthogonality of $\Phi_n(x)$, one can write ($x=\mathrm{e}^{\mathrm{i}\theta}$)

$$\tilde{Q}_n(z)=I_\mu\left\{\frac{z+x}{z-x}\left(\frac{z^p}{x^p}\tilde{\Phi}_{n-\alpha}(x)-\tilde{\Phi}_{n-\alpha}(z)\right)\right\}$$

p being an arbitrary nonnegative integer such that $0\leqslant p\leqslant n-\alpha-1$.

Furthermore,

$$\tilde{Q}_n(z) = I_\mu\left\{\frac{z+x}{z-x}\left(\frac{z^p}{x^p}\tilde{\Phi}_{n-\alpha}(x) - \tilde{\Phi}_{n-\alpha}(z)\right)\right\} = I_\mu\left\{\frac{z+x}{z-x}\left(\frac{z^p}{x^{p+\alpha}}\Phi_n(x) - \frac{1}{z^\alpha}\Phi_n(z)\right)\right\}$$

$$= \frac{1}{z^\alpha} I_\mu\left\{\frac{z+x}{z-x}\left(\frac{z^{p+\alpha}}{x^{p+\alpha}}\Phi_n(x) - \Phi_n(z)\right)\right\}, \quad 0 \leqslant p \leqslant n-\alpha-1.$$

Thus,

$$Q_n(z) = z^\alpha \tilde{Q}_n(z) = I_\mu\left\{\frac{z+x}{z-x}\left(\frac{z^{p+\alpha}}{x^{p+\alpha}}\Phi_n(x) - \Phi_n(z)\right)\right\}. \tag{4.8}$$

Since $0 \leqslant p \leqslant n-\alpha-1$, then $\alpha \leqslant p+\alpha \leqslant n-1$, $(0 \leqslant \alpha \leqslant n-1)$. Therefore, by Jones et al. [8, Theorem 4.1] it holds,

$$Q_n(z) = \Omega_n(z),$$

where $\Omega_n(z)$ is given as in formula (4.3). Hence, we have proved the following.

Proposition 4.2. *The rational function*

$$I_n(h(\bullet, z)) = \frac{\tilde{Q}_n(z)}{P_n(z)} = \frac{z^\alpha}{z^\alpha}\frac{\tilde{Q}_n(z)}{P_n(z)} = \frac{\Omega_n(z)}{\Phi_n(z)}$$

represents the weak (n, n) two-point Padé approximant of an order $(n, n+1)$ for the pair (L_0, L_∞). In the particular case $\Phi_n(0) \neq 0$, i.e., $\alpha = 0$, then, such a rational function represents the (n, n) two-point Padé approximant of an order $(n, n+1)$ for the pair (L_0, L_∞) in the strong sense.

In particular, if the zeros are all simple and $\alpha = 0$, then one has

$$A_{n,j} = -\frac{\Omega_n(x_j)}{2x_j\Phi_n'(x_j)}, \quad 1 \leqslant j \leqslant n. \tag{4.9}$$

Now, taking $p = 0$ in (4.8), it follows

$$\Omega_n(z) = \int_0^{2\pi} \frac{z+\mathrm{e}^{\mathrm{i}\theta}}{z-\mathrm{e}^{\mathrm{i}\theta}}(\Phi_n(\mathrm{e}^{\mathrm{i}\theta}) - \Phi_n(z))\,\mathrm{d}\mu(\theta).$$

Thus, for $1 \leqslant j \leqslant n$,

$$\Omega_n(x_j) = \int_0^{2\pi} \frac{x_j+\mathrm{e}^{\mathrm{i}\theta}}{x_j-\mathrm{e}^{\mathrm{i}\theta}}\Phi_n(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) = \frac{k_n}{2\pi}\int_0^{2\pi} \frac{[(x_j+\mathrm{e}^{\mathrm{i}\theta})/(x_j-\mathrm{e}^{\mathrm{i}\theta})]\Phi_n(\mathrm{e}^{\mathrm{i}\theta})}{|\Phi_n(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta$$

$$= \frac{k_n}{2\pi}\int_0^{2\pi} \frac{x_j+\mathrm{e}^{\mathrm{i}\theta}}{x_j-\mathrm{e}^{\mathrm{i}\theta}}\frac{e^{n\mathrm{i}\theta}}{\Phi_n^*(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\theta.$$

By the Residue Theorem

$$\Omega_n(x_j) = \frac{-2x_j^n k_n}{\Phi_n^*(x_j)}, \quad 1 \leqslant j \leqslant n.$$

Hence, by replacing in (4.9), one gets

$$A_{n,j} = -\frac{\Omega_n(x_j)}{2x_j\Phi_n'(x_j)} = \frac{k_n x_j^{n-1}}{\Phi_n'(x_j)\Phi_n^*(x_j)}, \quad 1 \leqslant j \leqslant n,$$

which coincides with formula (2.8).

Finally, it should be noted that there is another characterization of the sequences of monic orthogonal polynomials with respect to a measure (see [11] or [2]) in terms of the sequence $\{\Omega_n(z)\}$:

Proposition 4.3. *Let $\{\Phi_n\}$ and $\{\Omega_n\}$ be two sequences of polynomials with $\deg \Phi_n = n$. Then, $\{\Phi_n\}$ is the sequence of monic polynomial orthogonal with respect to a positive measure μ if and only if $\{\Phi_n\}$ and $\{\Omega_n\}$ satisfy*

$$\Phi_n(z)L_0(z) - \Omega_n(z) = O(z^n), \quad |z| < 1,$$

$$\Phi_n(z)L_\infty(z) - \Omega_n(z) = O\left(\frac{1}{z}\right), \quad |z| > 1,$$

where Ω_n is the associated Szegö polynomial given by formula (4.3) and L_0 and L_∞ are the expansions at 0 and ∞, respectively, of the Herglotz–Riesz transform with respect to μ.

The role played by the rational function $\Omega_n(z)/\Phi_n(z)$ will be clearly displayed in the next section.

5. Error estimates and convergence

We are now interested in providing different expressions of the error term $I_\mu(f) - I_n(f)$. For this purpose, consider first the function $h(x,z)$ as in formula (4.4). First, by (4.5), we have

$$\begin{aligned} I_n(h) &= \sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1} A_{m,j}h^{(j)}(x_m) = \sum_{m=1}^{s} A_{m,0}h(x_m) + \sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1} A_{m,j}h^{(j)}(x_m) \\ &= \sum_{m=1}^{s} A_{m,0}\frac{x_m+z}{x_m-z} + 2z\sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1}\frac{(-1)^j j! A_{m,j}}{(x_m-z)^{j+1}} = \frac{\Omega_n(z)}{\Phi_n(z)}. \end{aligned} \tag{5.1}$$

Observe that for $j \geqslant 1$

$$h^{(j)}(x_m) = 2z\frac{(-1)^j j!}{(x-z)^{j+1}}; \quad \left(\frac{x+z}{x-z} = 1 + \frac{2z}{x-z}\right).$$

Assume now that $f(z)$ is analytic in a bounded domain G containing $\bar{\mathbb{D}}$. Let Γ be the boundary of G. Assume $f(0) = 0$ (otherwise, we can work with the function $f(z) - f(0)$). Thus, by Cauchy and Fubini Theorem,

$$I_\mu(f) = \frac{1}{2\pi i}\int_\Gamma F_\mu(z)g(z)\,\mathrm{d}z \tag{5.2}$$

with $F_\mu(z)$ as in formula (4.3) and $g(z) = -f(z)/2z$ which is also analytic in G.

On the other hand,

$$I_n(f) = \sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1} A_{m,j} f^{(j)}(x_m) = \sum_{m=1}^{s} A_{m,0} f(x_m) + \sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1} A_{m,j} f^{(j)}(x_m)$$
$$= \sum_{m=1}^{s} A_{m,0} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{x_m+z}{x_m-z} g(z)\,\mathrm{d}z + \sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1} A_{m,j} f^{(j)}(x_m).$$

But $f^{(j)}(x_m) = (j!/2\pi\mathrm{i}) \int_{\Gamma} \dfrac{f(z)}{(z-x_m)^{j+1}}\,\mathrm{d}z$. Therefore,

$$I_n(f) = \sum_{m=1}^{s} A_{m,0} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{x_m+z}{x_m-z} g(z)\,\mathrm{d}z + \sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1} A_{m,j} \frac{j!}{2\pi\mathrm{i}} \int_{\Gamma} \frac{f(z)}{(z-x_m)^{j+1}}\,\mathrm{d}z$$
$$= \sum_{m=1}^{s} A_{m,0} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{x_m+z}{x_m-z} g(z)\,\mathrm{d}z - \sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1} A_{m,j} \frac{j!(-1)^j}{2\pi\mathrm{i}} \int_{\Gamma} \frac{f(z)}{(x_m-z)^{j+1}}\,\mathrm{d}z.$$

Thus,

$$I_n(f) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \left(\sum_{m=1}^{s} A_{m,0} \left(\frac{x_m+z}{x_m-z} \right) g(z) - \sum_{m=1}^{s}\sum_{j=1}^{\alpha_m-1} \frac{j!(-1)^j A_{m,j}}{(x_m-z)^{j+1}} f(z) \right) \mathrm{d}z$$

and by (5.1) one can finally deduce

$$I_n(f) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{\Omega_n(z)}{\Phi_n(z)} g(z)\,\mathrm{d}z. \tag{5.3}$$

Then, using (5.2) and (5.3), the following result has been proved:

Theorem 5.1. *Let $f(z)$ be an analytic function in a bounded domain G containing $\bar{\mathbb{D}}$ and let Γ be its boundary. Then*

$$I_\mu(f) - I_n(f) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \left(F_\mu(z) - \frac{\Omega_n(z)}{\Phi_n(z)} \right) g(z)\,\mathrm{d}z, \tag{5.4}$$

where $g(z) = f(z)/-2z$.

Thus, the error in the quadrature formula is essentially dominated by $F_\mu(z) - \Omega_n(z)/\Phi_n(z)$. It is known, ([8]), that the sequence $\{\Omega_n(z)/\Phi_n(z)\}$ converges uniformly to $F_\mu(z)$ on any compact in $\mathbb{E}$. Since the boundary Γ of the domain G defined above is contained in $\mathbb{E}$, then $I_n(f)$ converges to $I_\mu(f)$ with f as in Theorem 5.1.

In order to obtain estimates of the rate of convergence, consider now the Laurent polynomial L_n in $\Lambda_{-p,q}$; $(p+q = n-\alpha-1)$ interpolating f at the node x_m with multiplicity α_m, respectively, for all $m = 1,\ldots,s$. Then, we can write

$$L_n(z) = \sum_{m=1}^{s}\sum_{j=0}^{\alpha_m-1} \tilde{L}_{m,j}(z) f^{(j)}(x_m),$$

where $\tilde{L}_{m,j}(z)$ satisfies the conditions given by formula (4.1). One has, again by Theorem 4.1

$$I_n(f) = I_\mu(L_n)$$

and thus $I_\mu(f) - I_n(f) = I_\mu(f - L_n)$. Therefore, in this case, the error in the quadrature formula is dominated by the error in the interpolation. Let f be an analytic function on a bounded domain G containing $\bar{\mathbb{D}}$ and let Γ be its boundary. Then, by Walsh [14, p. 50],

$$f(z) - L_n(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{t}{z}\right)^p \frac{\tilde{\Phi}_{n-\alpha}(z)}{\tilde{\Phi}_{n-\alpha}(t)} \frac{f(t)}{t-z}\,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{t}{z}\right)^{p+\alpha} \frac{\Phi_n(z)}{\Phi_n(t)} \frac{f(t)}{t-z}\,\mathrm{d}t. \tag{5.5}$$

Now, the following is needed:

Let μ denote a finite Borel measure on $\mathbb{C}$ with compact support $S(\mu)$. Let Ω be the unbounded component of $\bar{\mathbb{C}} \backslash S(\mu)$ and $\partial\Omega$ its boundary, where $\bar{\mathbb{C}} = \bar{\mathbb{C}} \cup \{\infty\}$. Then,

Definition 5.2. A measure μ is said to be regular if the limit

$$\limsup_{n\to\infty} |\varphi_n(z)|^{1/n} = 1$$

holds true "quasi-everywhere" on $\partial\Omega$, where $\varphi_n(z)$ is the orthonormal polynomial with respect to μ.

For other equivalent definitions see [12, p. 60, Theorem 3.1.1].

Lemma 5.3 (see Stahl and Totik [12, p. 59]). *Let $\varphi_n(z)$ be the orthonormal polynomial with respect to a regular measure μ. Then*

$$\lim_{n\to\infty} |\varphi_n(z)|^{1/n} = |z| \tag{5.6}$$

uniformly on $\bar{\mathbb{E}}$.

Theorem 5.4. *Let f be analytic in a bounded domain G containing $\bar{\mathbb{D}}$ and let Γ be its boundary. If*

$$\lim_{n\to\infty} \frac{p(n) + \alpha(n)}{n} = s, \quad 0 < s < 1,$$

then

$$\lim_{n\to\infty} |I_\mu(f) - I_n(f)|^{1/n} = r(\Gamma, s) = \max_{t\in\Gamma} \left\{ \frac{1}{|t|^{1-s}} \right\} < 1.$$

Here $\alpha(n)$, $0 \leqslant \alpha(n) \leqslant n-1$ denotes the multiplicity of $z=0$ in $\Phi_n(z)$ and $\{p(n)\}$ is an arbitrary sequence of nonnegative integers such that $0 \leqslant p(n) \leqslant n - \alpha(n) - 1$.

Proof. Let $z \in \mathbb{T}$ and $t \in \Gamma$. Then, by (5.5)

$$\begin{aligned} |f(z) - L_n(z)| &\leqslant \max_{t\in\Gamma} |t|^{p(n)+\alpha(n)} \max_{t\in\Gamma} \left\{ \frac{1}{|t|-1} \right\} \frac{1}{2\pi} \int_\Gamma \left| \frac{\varphi_n(z)}{\varphi_n(t)} \right| |f(t)||\mathrm{d}t| \\ &= M(f) l(\Gamma) \max_{t\in\Gamma} |t|^{p(n)+\alpha(n)} \max_{t\in\Gamma} \left\{ \frac{1}{|t|-1} \right\} \max_{z\in T} |\varphi_n(z)| \max_{t\in\Gamma} \left| \frac{1}{\varphi_n(t)} \right|. \end{aligned}$$

where $M(f)=\max_{t\in\Gamma}|f(t)|$ and $l(\Gamma)=(1/2\pi)\int_\Gamma|\mathrm{d}t|$. On the other hand, since φ_n is a continuous function on $\mathbb{T}$, then $\max_{z\in\mathbb{T}}|\varphi_n(z)|=|\varphi_n(z_n)|$, $z_n\in\mathbb{T}$. Thus,

$$\left(\max_{z\in\mathbb{T}}|\varphi_n(z)|\right)^{1/n}=|\varphi_n(z_n)|^{1/n}\leqslant\max_{z\in\mathbb{T}}|\varphi_n(z)|^{1/n}.$$

Thus, by Lemma 5.3

$$\begin{aligned}\limsup_{n\to\infty}\left(\max_{z\in\mathbb{T}}|\varphi_n(z)|\right)^{1/n} &\leqslant \limsup_{n\to\infty}\max_{z\in\mathbb{T}}|\varphi_n(z)|^{1/n}\\ &\leqslant \max_{z\in\mathbb{T}}\limsup_{n\to\infty}|\varphi_n(z)|^{1/n}\\ &= \max_{z\in\mathbb{T}}\{|z|\}=1. \end{aligned}\tag{5.7}$$

Similarly,

$$\limsup_{n\to\infty}\left(\max_{t\in\Gamma}\left|\frac{1}{\varphi_n(t)}\right|\right)^{1/n}\leqslant\max_{t\in\Gamma}\left\{\frac{1}{\liminf_{n\to\infty}|\varphi_n(t)|^{1/n}}\right\}=\max_{t\in\Gamma}\left\{\frac{1}{|t|}\right\}.\tag{5.8}$$

Therefore, using (5.7) and (5.8),

$$\limsup_{n\to\infty}|f(z)-L_n(z)|^{1/n}\leqslant\max_{t\in\Gamma}\left\{\frac{1}{|t|^{1-s}}\right\}<1$$

and the proof follows. □

Next, let us see how this estimate of the rate of convergence can be improved. For it, we will obtain another expression of the error by considering the Hermite–Laurent interpolatory polynomial. Indeed, given the nodes x_m, $m=1,\dots,s$, $(x_m\neq 0)$ with multiplicity α_m, $m=1,\dots,s$, $(\sum_{m=1}^{s}\alpha_m=n-\alpha)$, then, there exists a unique Laurent polynomial $H_n\in\Lambda_{-(n-2\alpha-1),n}$ such that

$$H_n^{(j)}(x_m)=f^{(j)}(x_m);\ j=0,\dots,2\alpha_m-1;\ m=1,\dots,s.$$

We can write

$$H_n(x)=\sum_{m=1}^{s}\sum_{l=0}^{\alpha_m-1}H_{m,l}(x)f^{(l)}(x_m)+\sum_{m=1}^{s}\sum_{l=\alpha_m}^{2\alpha_m-1}\tilde{H}_{m,l}(x)f^{(l)}(x_m),$$

where $H_{m,l}$, $\tilde{H}_{m,l}\in\Lambda_{-(n-2\alpha-1),n}$. They satisfy the following conditions:

$$H_{m,l}^{(k)}(x_j)=\delta_{m,j}\delta_{l,k},\quad l=0,\dots,\alpha_m-1,$$

$$\tilde{H}_{m,l}^{(k)}(x_m)=\delta_{m,j}\delta_{l,k},\quad l=\alpha_m,\dots,2\alpha_m-1$$

for all $k=0,\dots,2\alpha_m-1$ and $m,j=1,2,\dots,s$. Since $\tilde{H}_{m,l}\in\Lambda_{-(n-2\alpha-1),n}$ and $\tilde{H}_{m,l}^{(k)}(x_m)=0$; $k=0,\dots,\alpha_m-1$, one has

$$\tilde{H}_{m,l}(x)=\frac{\Phi_n(x)}{x^{n-\alpha-1}}S(x),$$

where $S \in \mathbb{P}_{n-\alpha-1}$. Let $S(x) = \sum_{j=0}^{n-\alpha-1} a_j x^j$. Then

$$\frac{S(x)}{x^{n-\alpha-1}} = \sum_{j=0}^{n-\alpha-1} a_{n-\alpha-1-j} x^{-j}.$$

Thus,

$$\begin{aligned} I_\mu(\tilde{H}_{m,l}) &= \sum_{j=0}^{n-\alpha-1} a_{n-\alpha-1-j} \left(\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(x) x^{-j} \, \mathrm{d}\mu(\theta) \right) \\ &= \sum_{j=0}^{n-\alpha-1} a_{n-\alpha-1-j} (\Phi_n(z), z^j)_\mu = 0. \end{aligned}$$

Therefore,

$$I_\mu(\tilde{H}_{m,l}) = 0, \quad \forall l = \alpha_m, \ldots, 2\alpha_m - 1, \; m = 1, \ldots, s.$$

Now, define

$$\begin{aligned} \tilde{I}_n(f) = I_\mu(H_n) &= \sum_{m=1}^{s} \sum_{l=0}^{\alpha_m - 1} I_\mu(H_{m,l}) f^{(l)}(x_m) + \sum_{m=1}^{s} \sum_{l=\alpha_m s}^{2\alpha_m - 1} I_\mu(\tilde{H}_{m,l}) f^{(l)}(x_m) \\ &= \sum_{m=1}^{s} \sum_{l=0}^{\alpha_m - 1} I_\mu(H_{m,l}) f^{(l)}(x_m) = \sum_{m=1}^{s} \sum_{l=0}^{\alpha_m - 1} B_{l,m} f^{(l)}(x_m). \end{aligned}$$

Notice that $\tilde{I}_n(f)$ is exact in $\Lambda_{-(n-2\alpha-1),n}$ and so $I_\mu(f) = (1/2\pi) \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta}) \, \mathrm{d}\mu(\theta) = I_n(f) = \tilde{I}_n(f)$, that is,

$$\sum_{m=1}^{s} \sum_{l=0}^{\alpha_m - 1} A_{l,m} f^{(l)}(x_m) = \sum_{m=1}^{s} \sum_{l=0}^{\alpha_m - 1} B_{l,m} f^{(l)}(x_m).$$

In particular, if we take $f = H_{m,l}$, since $H_{m,l}^{(k)}(x_m) = \delta_{m,l}$; $k = 0, \ldots, \alpha_m - 1$, one has $A_{l,m} = B_{l,m}$; $\forall l = 0, \ldots, \alpha_m - 1$; $m = 1, \ldots, s$.

Therefore, we have proved the following

Proposition 5.5. *Let H_n be the unique Hermite–Laurent interpolatory polynomial in $\Lambda_{-(n-2\alpha-1),n}$ such that*

$$H_n^{(j)}(x_m) = f^{(j)}(x_m); \; j = 0, \ldots, 2\alpha_m - 1; \; m = 1, \ldots, s,$$

where x_m; $m = 1, \ldots, s$ are the nonvanishing zeros of the monic orthogonal polynomial $\Phi_n(z)$ with respect to the positive measure μ. Then

$$I_n(f) = I_\mu(H_n),$$

where as usual, $I_n(f)$ denotes the nth Gaussian quadrature formula given in Section 2.

From the above proposition one can obtain the following expression for the error in the quadrature formula:

$$I_\mu(f) - I_n(f) = I_\mu(f) - I_\mu(H_n) = I_\mu(f - H_n).$$

Let f be analytic in a bounded domain G containing $\bar{\mathbb{D}}$ and let Γ be its boundary. By Walsh [14] we have

$$\begin{aligned} f(z) - H_n(z) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{t}{z}\right)^{n-2\alpha-1} \frac{\tilde{\Phi}^2_{n-\alpha}(z)}{\tilde{\Phi}^2_{n-\alpha}(t)} \frac{f(t)}{t-z} \,\mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{t}{z}\right)^{n-1} \frac{\Phi^2_n(z)}{\Phi^2_n(t)} \frac{f(t)}{t-z} \,\mathrm{d}t. \end{aligned}$$

Therefore, by Fubini Theorem ($z = \mathrm{e}^{\mathrm{i}\theta}$)

$$\begin{aligned} I_\mu(f) - I_n(f) = I_\mu(f - H_n) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{t}{z}\right)^{n-1} \frac{\Phi^2_n(z)}{\Phi^2_n(t)} \frac{f(t)}{t-z} \,\mathrm{d}t \right) \mathrm{d}\mu(\theta) \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{t^{n-1}}{\Phi^2_n(t)} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi^2_n(z)}{z^{n-1}(t-z)} \,\mathrm{d}\mu(\theta) \right) f(t) \,\mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{-2t^n}{\Phi^2_n(t)} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi^2_n(z)}{z^{n-1}(t-z)} \,\mathrm{d}\mu(\theta) \right) \frac{f(t)}{-2t} \,\mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{2t^n}{\Phi^2_n(t)} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi^2_n(z)}{z^{n-1}(z-t)} \,\mathrm{d}\mu(\theta) \right) g(t) \,\mathrm{d}t. \end{aligned} \tag{5.9}$$

Remark 5.6. *By* (5.4) *and* (5.9) *we have obtained, for any* $z \in \mathbb{E}$, *the following integral representation of the error for the two-point Padé approximant:*

$$F_\mu(z) - \frac{\Omega_n(z)}{\Phi_n(z)} = \frac{2z^n}{\Phi^2_n(z)} \frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi^2_n(x)}{x^{n-1}(x-z)} \,\mathrm{d}\mu(\theta) \quad (x = \mathrm{e}^{\mathrm{i}\theta}), \tag{5.10}$$

where Ω_n *is defined as in formula* (4.3). *Observe that formula* (5.10) *was already obtained in* [3] *by a different method.*

Similarly to Theorem 5.4 but without the condition $\lim_{n\to\infty}(p(n) + \alpha(n))/n = s,\ 0 < s < 1$, one can prove the following

Theorem 5.7. *Let* f *be an analytic function in a bounded domain* G *containing* $\bar{\mathbb{D}}$ *and let* Γ *be its boundary. Then, if* H_n *is the Laurent polynomial defined above, one has*

$$\limsup_{n\to\infty} |f(z) - H_n(z)|^{1/n} \leqslant \max_{t\in\Gamma} \left\{ \frac{1}{|t|} \right\} < 1$$

uniformly on $\mathbb{T}$. *Consequently,* $\limsup_{n\to\infty} |I_\mu(f) - I_n(f)|^{1/n} = r(\Gamma) = \max_{t\in\Gamma}\{1/|t|\} < 1.$

Remark 5.8. If f is an entire function, then we can choose a value of $|t|$ large enough in such a way that $\lim_{n\to\infty}|I_\mu(f)-I_n(f)|^{1/n}=0$ holds.

In short, for an analytic function on a neighborhood of $\bar{\mathbb{D}}$, we have proved that the Gaussian quadrature formulae converge geometrically. Next, we will consider, in a first step, functions f which are analytic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$, and in a second step, functions f analytic on D and integrable on $\bar{\mathbb{D}}$. In both cases the convergence of the quadrature formula is proved but without any estimation of the rate of convergence.

Indeed, let $\Phi_n(z)$ be the monic Szegö polynomial with respect to a probability measure μ as in (2.3) and satisfying (2.4). Then we have the following

Proposition 5.9. *Let* $\mu_k=(1/2\pi)\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\mu(\theta)$, $k\in\mathbb{Z}$ *be the normalized moments with respect to the positive measure* μ. *Let* f *be analytic in* $\mathbb{D}$ *and continuous in* $\bar{\mathbb{D}}$. *Then,*

$$\lim_{n\to\infty}I_n(f)=I_\mu(f).$$

Proof. Since f is analytic in $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$, then $\forall\varepsilon>0,\exists P\in\mathbb{P}:|f(x)-P(x)|<\varepsilon,\ \forall x\in\mathbb{T}$. Let $P(z)=\sum_{j=0}^{N}p_jz^j$. Then, for all $n\geqslant N$ we have

$$|I_\mu(f)-I_n(f)|=|I_\mu(f)-I_\mu(P)+I_\mu(P)-I_n(f)|\leqslant|I_\mu(f-P)|+|I_n(P-f)|.$$

We have (recall that $\mu_0=(1/2\pi)\int_0^{2\pi}\mathrm{d}\mu(\theta)=1$)

$$|I_\mu(f-P)|=\left|\frac{1}{2\pi}\int_0^{2\pi}(f(\mathrm{e}^{\mathrm{i}\theta})-P(\mathrm{e}^{\mathrm{i}\theta}))\,\mathrm{d}\mu(\theta)\right|\leqslant\frac{1}{2\pi}\int_0^{2\pi}|f(\mathrm{e}^{\mathrm{i}\theta})-P(\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\mu(\theta)<\varepsilon.$$

On the other hand,

$$\begin{aligned}|I_n(P-f)|&=\frac{|k_n|}{2\pi}\left|\int_0^{2\pi}\frac{f(\mathrm{e}^{\mathrm{i}\theta})-P(\mathrm{e}^{\mathrm{i}\theta})}{|\Phi_n(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta\right|\leqslant\frac{k_n}{2\pi}\int_0^{2\pi}\frac{|f(\mathrm{e}^{\mathrm{i}\theta})-P(\mathrm{e}^{\mathrm{i}\theta})|}{|\Phi_n(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta\\&<\varepsilon\frac{k_n}{2\pi}\int_0^{2\pi}\frac{1}{|\Phi_n(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta=\varepsilon.\end{aligned}$$

Therefore,

$$|I_\mu(f)-I_n(f)|<2\varepsilon.\qquad\square$$

In the above conditions, we can obtain a similar result for integrable functions. In this case the following result will be used.

Lemma 5.10 (See Szegö [13, p. 11, Theorem 1.5.4]). *Let* f *be a bounded real valued function on* $[0,2\pi]$, $\mu(\theta)$ *a positive measure and let the Riemann–Stieltjes integral* $\int_0^{2\pi}f(\theta)\,\mathrm{d}\mu(\theta)$ *exist. Then, for every* $\varepsilon>0$, *there exist trigonometric polynomials* $p(\theta)$ *and* $P(\theta)$ *such that*

$$\inf f-\varepsilon\leqslant p(\theta)\leqslant f(\theta)\leqslant P(\theta)\leqslant\sup f+\varepsilon\quad\textit{and}\quad\frac{1}{2\pi}\int_0^{2\pi}(P(\theta)-p(\theta))\,\mathrm{d}\mu(\theta)<\varepsilon.$$

Proposition 5.11. *Let f be analytic in D and integrable in $\bar{\mathbb{D}}$. Then, if* $\lim_{n\to\infty}(n-\alpha(n))=\infty$

$$\lim_{n\to\infty} I_n(f)=I_\mu(f).$$

Proof. We can write $f(e^{i\theta})=f_1(\theta)+if_2(\theta))$. f_1 and f_2 are two real valued functions on $[0,2\pi]$. Then $I_\mu(f)=I_\mu(f_1)+iI_\mu(f_2)$.

Let $\varepsilon>0$. Then, by Lemma 5.10, there exist trigonometric polynomials $p(\theta)$ and $P(\theta)$ such that

$$p(\theta)\leqslant f_1(\theta)\leqslant P(\theta) \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi}(P(\theta)-p(\theta))\,d\mu(\theta)<\varepsilon. \tag{5.11}$$

Taking into account

$$\cos k\theta=\frac{e^{ik\theta}+e^{-ik\theta}}{2}, \quad \sin k\theta=\frac{e^{ik\theta}-e^{-ik\theta}}{2i},$$

one has ($z=e^{i\theta}$)

$$p(\theta)=\sum_{k=-N}^{N}\alpha_k z^k=T_N(z)\in\Lambda_{-N,N}, \quad \text{and} \quad P(\theta)=\sum_{k=-M}^{M}\beta_k z^k=T_M(z)\in\Lambda_{-M,M}.$$

If $n\geqslant\max\{M+\alpha+1,N+\alpha+1\}$, from (5.11), we get

$$\frac{1}{2\pi}\int_0^{2\pi}(T_M(e^{i\theta})-f_1(\theta))\,d\mu(\theta)<\varepsilon \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi}(f_1(\theta)-T_N(e^{i\theta}))\,d\mu(\theta)<\varepsilon.$$

Hence,

$$\begin{aligned}
I_\mu(f_1)-\varepsilon &< \frac{1}{2\pi}\int_0^{2\pi}f_1(\theta)\,d\mu(\theta)-\frac{1}{2\pi}\int_0^{2\pi}(f_1(\theta)-T_N(e^{i\theta}))\,d\mu(\theta)\\
&= I_\mu(T_N)=I_n(T_N)=\frac{k_n}{2\pi}\int_0^{2\pi}\frac{T_N(e^{i\theta})}{|\Phi_n(e^{i\theta})|^2}\,d\theta\\
&< \frac{k_n}{2\pi}\int_0^{2\pi}\frac{f_1(\theta)}{|\Phi_n(e^{i\theta})|^2}\,d\theta<\frac{k_n}{2\pi}\int_0^{2\pi}\frac{T_M(e^{i\theta})}{|\Phi_n(e^{i\theta})|^2}\,d\theta.
\end{aligned}$$

Therefore, since $\Re(I_n(f))=(k_n/2\pi)\int_0^{2\pi}(f_1(\theta)/|\Phi_n(e^{i\theta})|^2)\,d\theta$, one has

$$I_\mu(f_1)-\varepsilon<I_n(T_N)<\Re(I_n(f))<I_n(T_M)=I_\mu(T_M)=I_\mu(f_1)+I_\mu(T_M-f_1)=I_\mu(f_1)+\varepsilon.$$

Thus,

$$|I_\mu(f_1)-\Re(I_n(f))|<\varepsilon. \tag{5.12}$$

Similarly,

$$|I_\mu(f_2)-\Re(I_n(f))|<\varepsilon. \tag{5.13}$$

Finally, from (5.12) and (5.13) the proof follows. □

Remark 5.12. Observe that Proposition 5.11 is still true for every function f only integrable on $\mathbb{T}$. Furthermore, it is known, (see [5]), that Proposition 5.9 is still valid for every continuous function

f on $\mathbb{T}$. However, we should use in both cases, as a definition of $I_n(f)$, integral (2.2) since now $I_n(f)$ as given by (1.1) becomes meaningless in general.

6. Numerical results

We will consider the following positive measures $\mathrm{d}\mu_1(\theta) = [(1 + \cos\theta)/2\pi]\,\mathrm{d}\theta$, $\mathrm{d}\mu_2(\theta) = [(1 - \cos\theta)/2\pi]\,\mathrm{d}\theta$ and $\mathrm{d}\mu_3(\theta) = (\sin^2\theta/2\pi)\,\mathrm{d}\theta$.

For $n = 8$ we have deduced the nodes $\{x_j\}_{j=1}^{8}$ and coefficients $\{A_j\}_{j=1}^{8}$ with respect to the above measures (see Table 1).

Let

$$f_1(z) = \frac{\sin(z)}{1.5 - z}, \quad f_2(z) = \frac{\sin(z)}{4 - z}, \quad f_3(z) = \frac{\sin(z)}{10 - z}.$$

Table 1

Nodes	Coefficients
$\mu_1(\theta)$	
$x_1 = -0\ 567289 - 0.570292I$	$A_1 = 0.0284838 + 0.0204506I$
$x_2 = -0\ 567289 + 0.570292I$	$A_2 = 0.0284838 - 0.0204506I$
$x_3 = -0.0792246 - 0.758031I$	$A_3 = 0.0940173 + 0.0345958I$
$x_4 = -0.0792246 + 0.758031I$	$A_4 = 0.0940173 - 0.0345958I$
$x_5 = 0.398567 - 0.625602I$	$A_5 = 0.165726 + 0.0318851I$
$x_6 = 0.398567 + 0.625602I$	$A_6 = 0.165726 - 0.0318851I$
$x_7 = 0.692391 - 0.24052I$	$A_7 = 0.211773 + 0.0128452I$
$x_8 = 0.692391 + 0.24052I$	$A_8 = 0.211773 - 0.0128452I$
$\mu_2(\theta)$	
$x_1 = -0.692391 - 0.24052I$	$A_1 = 0.211773 - 0.0128452I$
$x_2 = -0.692391 + 0.24052I$	$A_2 = 0.211773 + 0.0128452I$
$x_3 = -0.398567 - 0.625602I$	$A_3 = 0.165726 - 0.0318851I$
$x_4 = -0.398567 + 0.625602I$	$A_4 = 0.165726 + 0.0318851I$
$x_5 = 0.0792246 - 0.758031I$	$A_5 = 0.0940173 - 0.0345958I$
$x_6 = 0.0792246 + 0.758031I$	$A_6 = 0.0940173 + 0.0345958I$
$x_7 = 0.567289 - 0.570292I$	$A_7 = 0.0284838 - 0.0204506I$
$x_8 = 0.567289 + 0.570292I$	$A_8 = 0.0284838 + 0.0204506I$
$\mu_3(\theta)$	
$x_1 = -0.644147 - 0.526397I$	$A_1 = 0.0357665 + 0.0212742I$
$x_2 = -0.644147 + 0.526397I$	$A_2 = 0.0357665 - 0.0212742I$
$x_3 = 0.644147 - 0.526397I$	$A_3 = 0.0357665 - 0.0212742I$
$x_4 = 0.644147 + 0.526397I$	$A_4 = 0.0357665 + 0.0212742I$
$x_5 = 0.232822 + 0.76944I$	$A_5 = 0.0892335 + 0.0139831I$
$x_6 = 0.232822 - 0.76944I$	$A_6 = 0.0892335 - 0.0139831I$
$x_7 = -0\ 232822 + 0.76944I$	$A_7 = 0.0892335 - 0.0139831I$
$x_8 = -0\ 232822 - 0.76944I$	$A_8 = 0.0892335 + 0.0139831I$

Table 2

Node n	Exact error (Szegö) μ_1	Exact error (Gauss) μ_1
4	$1.576456E-01$	$1.342812E-02$
6	$7.669225E-02$	$4.368393E-03$
8	$3.512272E-02$	$1.532380E-03$

Table 3

Node n	Exact error (Szegö) μ_2	Exact error (Gauss) μ_2
4	$4.261750E-03$	$4.485824E-03$
6	$3.543889E-03$	$1.267323E-03$
8	$1.478630E-03$	$4.088199E-04$

Table 4

Node n	Exact error (Szegö) μ_3	Exact error (Gauss) μ_3
4	$9.039074E-03$	$3.587872E-03$
6	$4.737054E-03$	$1.115100E-03$
8	$2.036738E-03$	$3.800699E-04$

Table 5

Node n	Exact error (Szegö) μ_1	Exact error (Gauss) μ_1
4	$2.74201E-03$	$5.46331E-05$
6	$3.32082E-05$	$1.96088E-06$
8	$3.34996E-06$	$5.34858E-08$

For these functions, the exact error of Gaussian and Szegö quadrature formulae with respect to the measures $\mu_1(\theta)$, $\mu_2(\theta)$, and $\mu_3(\theta)$, respectively, are computed. In each case, and for $n = 4, 6, 8$, the zeros of the Szegö polynomial are all simple and different from zero, i.e., $\alpha = 0$ and the zeros of $\tilde{\Phi}_{n-\alpha}(z) = \Phi_n(z)$ are simple. For the function $f_1(z)$ the results are displayed in Tables 2–4.

For $f_2(z)$ the results are displayed in Tables 5–7.

Finally, for the function $f_3(z)$ the results are given in Tables 8–10.

From these tables we can see that the Gaussian formulae give better results than the Szegö formulae and that both quadrature formulae depend on the location of the singularities of the integrand $f(z)$ with respect to the unit circle.

Finally, it should be said that in [9] making use of certain continued fractions, computable error bound for the Szegö quadrature formulae can be deduced (see also [4]). In this respect, a similar

Table 6

Node n	Exact error (Szegö) μ_2	Exact error (Gauss) μ_2
4	$3.65415E-03$	$3.6313E-05$
6	$7.54454E-05$	$1.3048E-06$
8	$2.0878E-06$	$2.10162E-09$

Table 7

Node n	Exact error (Szegö) μ_3	Exact error (Gauss) μ_3
4	$1.68592E-03$	$9.79090E-06$
6	$3.10834E-05$	$3.30104E-07$
8	$1.24397E-06$	$5.46841E-09$

Table 8

Node n	Exact error (Szegö) μ_1	Exact error (Gauss) μ_1
4	$7.291970E-05$	$7.302898E-05$
6	$2.010499E-05$	$1.065462E-06$
8	$5.016616E-07$	$3.728045E-08$

Table 9

Node n	Exact error (Szegö) μ_2	Exact error (Gauss) μ_2
4	$1.426109E-03$	$6.218423E-05$
6	$4.625420E-05$	$9.036871E-07$
8	$7.913159E-07$	$2.536029E-08$

Table 10

Node n	Exact error (Szegö) μ_3	Exact error (Gauss) μ_3
4	$4.258231E-04$	$5.712364E-06$
6	$1.757401E-05$	$8.372155E-08$
8	$3.341328E-07$	$1.279191E-09$

treatment could be also carried out in order to obtain error bounds for the Gaussian quadratures considered here.

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